

# Non abelian bosonisation in three dimensional field theory

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## Abstract

We develop a method based on the generalised Stückelberg prescription for discussing bosonisation in the low energy regime of the SU(2) massive Thirring model in 2+1 dimensions. For arbitrary values of the coupling parameter the bosonised theory is found to be a nonabelian gauge theory whose physical sector is explicitly obtained. In the case of vanishing coupling this gauge theory can be identified with the SU(2) Yang-Mills Chern-Simons theory in the limit when the Yang-Mills term vanishes. Bosonisation identities for the fermionic current are derived.

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The conventional ideas [1] of bosonising a (1+1) dimensional fermionic theory have been recently extended in several directions. These include a wide variety of topics like ‘smooth bosonisation’ [2, 3, 4], connection between dual transformations and bosonisation [5, 6], bosonisation in higher dimensions [6, 7, 8, 9, 10, 11] etc. Most of these issues, however, are analysed in the context of abelian models. Thus, inspite of this recent plethora of papers, not much insight has been gained concerning nonabelian bosonisation, particularly in dimensions greater than (1+1). There is a paper [10] which discussed the bosonisation of the three-dimensional nonabelian massive Thirring model (in the low energy regime) but definitive results were obtained only in the weak coupling limit in which case the original model reduced to a theory of free massive fermions.

In this paper we develop a general formalism of discussing bosonisation in the low energy sector of nonabelian fermionic models in higher dimensions. To pinpoint our analysis we consider the bosonisation of  $SU(2)$  massive Thirring model in 2+1 dimensions. At the end it will be evident that considering a larger group of transformations (say  $SU(3)$ ) or increasing the dimensionality of space-time (say 3+1) are mere technical details and do not pose conceptual problems. Exactly as happened in the abelian model [8, 9, 11], the nonabelian Thirring model in the leading  $m^{-1}$  approximation gets mapped on to a gauge theory, but now it is nonabelian instead of being abelian. The physical sector of this gauge theory, for an arbitrary Thirring coupling, is abstracted by obtaining both the Gauss operator that generates the nonabelian gauge transformations and the physical hamiltonian. The weak coupling limit is then critically examined. It is shown that in this limit, a covariant combination of fields in the gauge theory simulating the Thirring model can be identified with the components of the field tensor occuring in the Yang-Mills-Chern-Simons (YMCS) theory [12] in the limit when the Yang-Mills term vanishes, such that the physical hamiltonians in the two gauge theories get mapped on to each other. These results provide a concrete hamiltonian realisation of the lagrangian approach [10] where the free massive theory was identified with the nonabelian Chern-Simons theory. Bosonisation identities for the fermionic currents are also provided.

It is worthwhile to say a few words about our methodology. A completely different approach than is usually followed for discussing abelian bosonisation [5, 6, 7, 8, 9, 11] will be adopted. In the abelian case master lagrangians [6, 8, 9, 11, 13] are suggested that interpolate between fermionic and bosonic

theories. Such a strategy, however, is ineffective for obtaining equivalences among nonabelian theories [10, 14]. Indeed, as stated earlier, the paper [10] that discussed bosonisation of the massive Thirring model through this technique yielded explicit results only in the weak coupling limit. Here we follow a generalised Stückelberg like embedding prescription [15] for converting second class systems into first class (gauge) systems. The Thirring model, in the low energy sector, is an example of a second class system thereby enabling us to exploit the Stückelberg procedure.

The partition function for the SU(2) massive Thirring model is given by,

$$Z = \int d[\psi, \bar{\psi}] \exp i \int d^3x (\bar{\psi}(i\rlap{\not{\partial}} + m)\psi - \frac{g^2}{2} j_\mu^a j^{\mu a}) \quad (1)$$

where,

$$j_\mu^a = \bar{\psi} \gamma_\mu \sigma^a \psi \quad (2)$$

is the fermionic current defined by the standard representation of the Pauli ( $\sigma^a$ ) - matrices. As usual, the four-fermion interaction can be simplified by introducing an auxiliary vector field  $B_\mu$ ,

$$Z = \int d[\psi, \bar{\psi}, B_\mu] \exp i \int d^3x (\bar{\psi}(i\rlap{\not{\partial}} + m + \rlap{\not{B}})\psi + \frac{1}{2g^2} B^{\mu a} B_\mu^a) \quad (3)$$

The integration over the fermion fields reduces to the familiar problem of computing the functional determinant in the presence of an external field. In general, this expression is nonlocal. Under some approximation (like the  $\frac{1}{m}$  expansion) it yields a local form. In this low energy regime the leading term is just the nonabelian Chern Simons term so that upto  $O(m^{-1})$  [10],

$$Z \approx \int dB_\mu \exp i \int d^3x (\frac{1}{8\pi} \epsilon^{\mu\nu\lambda} (B_\mu^a \partial_\nu B_\lambda^a + \frac{\epsilon^{abc}}{3} B_\mu^a B_\nu^b B_\lambda^c) + \frac{1}{2g^2} B^{\mu a} B_\mu^a) \quad (4)$$

Eq.(4) represents the partition function<sup>3</sup> of the nonabelian version of a self dual model considered in [16, 17, 18, 13]. In the abelian case it was possible to exploit the equivalence [17, 18, 13] between the self-dual model and the Maxwell-Chern-Simons theory to obtain a correspondance between the latter and the massive Thirring model. The above stated equivalence was demonstrated using either master Lagrangians [17, 18, 13, 8] or embedding

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<sup>3</sup>There is a technical issue which has been clarified in the discussion below (12)

prescription [13]. Since the master Lagrangian approach is ineffective for non-abelian theories [14, 10], we adopt the embedding approach.

The lagrangian obtainable from (4) is,

$$\mathcal{L} = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}(B_\mu^a\partial_\nu B_\lambda^a + \frac{1}{3}\epsilon^{abc}B_\mu^a B_\nu^b B_\lambda^c) + \frac{1}{2g^2}B_\mu^a B^{\mu a} \quad (5)$$

The equations of motion are given by,

$$\frac{1}{g^2}B_\mu^a + \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}F^{\nu\lambda,a} = 0 \quad (6)$$

with,

$$F_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + \epsilon^{abc}B_\mu^b B_\nu^c \quad (7)$$

It is easily seen that the  $\mu = 0$  component of (6) is an equation of constraint,

$$\chi^a = \frac{1}{g^2}B_0^a + \frac{1}{8\pi}\epsilon_{ij}F^{ij,a} \approx 0 \quad (8)$$

where the weak equality is to be interpreted in the sense of Dirac [19]. it is clear, therefore, that we are working with a constrained system. It is simple to extract and classify all the constraints. The momenta conjugate to  $B_\mu$  are given by,

$$\pi_0^a = \frac{\partial\mathcal{L}}{\partial\dot{B}^{0,a}} \approx 0, \quad \pi_i^a = \frac{\partial\mathcal{L}}{\partial\dot{B}^{i,a}} = \frac{1}{8\pi}\epsilon_{ij}B^{j,a} \quad (9)$$

which are the primary constraints. The second pair of constraints is just a manifestation of the symplectic structure of the Chern-Simons term. It is eliminated either by computing the Dirac brackets [19] or following the symplectic procedure [20], both of which yield the modified Poisson brackets

$$\{B_i^a(x), B_j^b(x')\} = 4\pi\delta^{ab}\epsilon_{ij}\delta(x - x') \quad (10)$$

The canonical hamiltonian obtained by a formal Legendre transform of (5) is found to be,

$$H_c = \int d^2x [\frac{1}{2g^2}(B_i^a B_i^a + B_0^a B_0^a) - B_0^a \chi^a] \quad (11)$$

where  $\chi^a$  is defined in (8). Time conserving the primary constraint  $\pi_0^a \approx 0$  just yields (8) as a secondary constraint. The constraint obtained in the

Lagrangian approach is thereby reproduced in the hamiltonian framework. Furthermore since,

$$\{\pi_0^a(x), \chi^b(x')\} = -\frac{1}{g^2}\delta^{ab}\delta(x-x') \not\approx 0 \quad (12)$$

these form a pair of second class constraints. No more constraints are therefore generated by Dirac's [19] iterative prescription. It is not surprising that we have obtained a second class system since the mass term in (5) breaks the gauge invariance. At this point we remark that the Lagrangian path integral for (5) should contain the constraint (8) as a delta functional in the measure [21]. However since  $B_0^a$  occurs quadratically in (5) the result of integrating out  $B_0^a$  is the same irrespective of the presence or absence of the delta functional [18]. It is for this reason that although the delta functional does not appear in (4), it can still be regarded as the effective action for the (nonabelian) self dual model (5).

The next step is to convert the second class system (5) into a true gauge (first class) system. A viable approach is to use the generalised Stückelberg formalism of Kunimasa and Goto [15]. The idea is to introduce extra fields  $\omega_\mu^a$  such that  $B_\mu^a + \omega_\mu^a$  is gauge covariant. In that case the embedded lagrangian,

$$\tilde{\mathcal{L}} = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}(B_\mu^a\partial_\nu B_\lambda^a + \frac{1}{3}\epsilon^{abc}B_\mu^a B_\nu^b B_\lambda^c) + \frac{1}{2g^2}(B_\mu^a + \omega_\mu^a)(B^{\mu,a} + \omega^{\mu,a}) \quad (13)$$

would be gauge invariant. As discussed in [15]  $\omega_\mu$  can be constructed from a  $2 \times 2$  unitary unimodular matrix  $M$  which transforms as

$$M \rightarrow MS \quad (14)$$

where  $S$  is the C-number unitary unimodular matrix that defines the gauge transformation,

$$B_\mu \rightarrow B'_\mu = S^{-1}B_\mu S + iS^{-1}\partial_\mu S. \quad (15)$$

Taking  $\omega_\mu$  to be,

$$\omega_\mu = -iM^{-1}\partial_\mu M \quad (16)$$

yields the transformed  $\omega_\mu$

$$\omega_\mu \rightarrow \omega'_\mu = S^{-1}\omega_\mu S - iS^{-1}\partial_\mu S \quad (17)$$

Combining (15) and (17) shows that  $B_\mu + \omega_\mu$  transforms covariantly,

$$B_\mu + \omega_\mu \rightarrow B'_\mu + \omega'_\mu = S^{-1}(B_\mu + \omega_\mu)S \quad (18)$$

as required to make (13) gauge invariant. A particular form for  $M$  is given [15] as a function of the Euler angles  $\psi, \theta, \phi$  ;

$$M = \exp(i\psi\sigma^3) \exp(i\theta\sigma^1) \exp(i\phi\sigma^3) \quad (19)$$

so that inserting in (16) we obtain,

$$\omega_\mu^1 = \cos \phi \partial_\mu \theta + \sin \theta \sin \phi \partial_\mu \psi \quad (20)$$

$$\omega_\mu^2 = \sin \phi \partial_\mu \theta - \sin \theta \cos \phi \partial_\mu \psi \quad (21)$$

$$\omega_\mu^3 = \partial_\mu \phi + \cos \theta \partial_\mu \psi \quad (22)$$

Note that the Euler angles are to interpreted as field variables. With (20-22) defining  $\omega_\mu^a$ , the Lagrangian (13) is gauge invariant. This is now explicitly revealed in the hamiltonian formalism. Apart from (9) the other canonical momenta are,

$$\pi_\theta = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\theta}} = \frac{1}{g^2} [(B_0^1 + \omega_0^1) \cos \phi + (B_0^2 + \omega_0^2) \sin \phi] \quad (23)$$

$$\begin{aligned} \pi_\psi &= \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\psi}} \\ &= \frac{1}{g^2} [B_0^1 + \omega_0^1) \sin \theta \sin \phi - (B_0^2 + \omega_0^2) \sin \theta \cos \phi \\ &\quad + (B_0^3 + \omega_0^3) \cos \theta] \end{aligned} \quad (24)$$

$$\pi_\phi = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\phi}} = \frac{1}{g^2} (B_0^3 + \omega_0^3) \quad (25)$$

Substituting the value of  $(B_0^3 + \omega_0^3)$  from (25) in (24) and then using (23) it is possible to express the covariant combinations in terms of phase space variables,

$$B_0^1 + \omega_0^1 = g^2 [\pi_\theta \cos \phi + \frac{\sin \phi}{\sin \theta} (\pi_\psi - \pi_\phi \cos \theta)] = g^2 L^1 \quad (26)$$

$$B_0^2 + \omega_0^2 = g^2 [\pi_\theta \sin \phi - \frac{\cos \phi}{\sin \theta} (\pi_\psi - \pi_\phi \cos \theta)] = g^2 L^2 \quad (27)$$

$$B_0^3 + \omega_0^3 = g^2 \pi_\phi = g^2 L^3 \quad (28)$$

Using the basic Poisson brackets,

$$\{\theta, \pi_\theta\} = \{\phi, \pi_\phi\} = \{\psi, \pi_\psi\} = \delta(x - x') \quad (29)$$

where the arguments of the fields have been suppressed, it is simple to show that  $L^a$  defined in (26 - 28) satisfy the angular momentum algebra,

$$\{L^a(x), L^b(x')\} = \epsilon^{abc} L^c \delta(x - x'). \quad (30)$$

The canonical hamiltonian is given by,

$$\tilde{H}_c = \int (\pi_0 B^0 + \pi_i B^i + \pi_\theta \dot{\theta} + \pi_\psi \dot{\psi} + \pi_\phi \dot{\phi} - \tilde{\mathcal{L}}) d^2x \quad (31)$$

Using (20 - 22) in conjunction with (26 - 28) for eliminating velocities, we obtain,

$$\tilde{H}_c = \int d^2x \left[ \frac{1}{2g^2} (B_i^a + \omega_i^a)^2 + \frac{g^2}{2} L^a L^a - B_0^a G^a \right] \quad (32)$$

where,

$$G^a = \frac{1}{8\pi} \epsilon_{ij} F_{ij}^a + L^a \quad (33)$$

and satisfies an algebra analogous to (30),

$$\{G^a(x), G^b(x')\} = \epsilon^{abc} G^c \delta(x - x') \quad (34)$$

where use has been made of (10). We also exploit the fact that the new fields and their conjugate momenta have vanishing brackets with the original phase space variables  $B_\mu, \pi^\mu$ . Time conserving the primary constraint  $\pi_0^a \approx 0$  now yields the secondary constraint,

$$G^a \approx 0 \quad (35)$$

In contrast to (12),  $\pi_0^a$  has vanishing brackets with  $G^b$ . The involutive algebra satisfied by  $G^a$  is already given in (34) so that the set of constraints  $\pi_0^a, G^a$  is first class, as expected. The canonical hamiltonian (32) is further simplified on the constraint surface (35) to yield the physical hamiltonian [22],

$$\tilde{H}_P = \int d^2x \left[ \frac{1}{2g^2} (B_i^a + \omega_i^a)^2 + \frac{g^2}{64\pi^2} F_{ij}^a F_{ij}^a \right] \quad (36)$$

In the hamiltonian formalism the first class constraint (35) plays the role of the Gauss operator and hence is the generator of time independent gauge transformations. The covariant transformation law (18) is thereby easily verified,

$$\{G^a(x), B_i^b + \omega_i^b(x')\} = \epsilon^{abc}(B_i^c + \omega_i^c)\delta(x - x') \quad (37)$$

where we have used (10) and,

$$\{L^a(x), \omega_i^b(x')\} = \epsilon^{abc}\omega_i^c\delta(x - x') + \delta^{ab}\partial_i\delta(x - x') \quad (38)$$

to compute the basic brackets. The  $\mu = 0$  component of (18) follows trivially on using (30). This completes the conversion of the second class system (5) into first class.

We therefore conclude that the Thirring model bosonises to a gauge theory whose physical sector is defined by the Gauss constraint (35) and the hamiltonian (36). All brackets are canonical except for (10) in which  $B_1, B_2$  are to be regarded as the canonical pair. The bosonisation achieved here is exact in the sense that the weak coupling limit is not imposed, as was an essential prerequisite in [10].

Let us next try to understand the implications of our construction and see whether it is possible to relate it to some known gauge theory. The second term in (36) resembles the familiar Yang-Mills contribution while the first involves the new fields. It is clear, therefore, that (36) cannot represent the pure Yang-Mills theory. In 2+1 dimensions it is possible to think of another familiar gauge theory - the YMCS theory [12] whose dynamics is governed by the lagrangian density,

$$\mathcal{L} = \frac{g^2}{32\pi^2} G_{\mu\nu}^a G^{\mu\nu,a} - \frac{1}{8\pi} \epsilon^{\mu\nu\rho} (A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3} \epsilon^{abc} A_\mu^a A_\nu^b A_\rho^c) \quad (39)$$

where  $G_{\mu\nu}^a$  is the field tensor (7) expressed in terms of the  $A_\mu$  field and, for reasons of comparison, we have taken the same coupling parameter  $g$  that appeared in the Thirring model (1). This theory is known to be first class with the constraints,

$$\pi_0^a \approx 0, \quad \Lambda^a = D_i \pi^{i,a} - \frac{1}{8\pi} \epsilon_{ij} \partial_i A_j^a \approx 0 \quad (40)$$

where  $\pi_\mu^a$  is the momentum conjugate to  $A^{\mu,a}$ ,

$$\pi_i^a = -\frac{g^2}{16\pi^2} G_{0i}^a - \frac{1}{8\pi} \epsilon_{ij} A^{j,a} \quad (41)$$

The constraint  $\Lambda^a \approx 0$  is the generator of gauge transformations enforced by the Lagrange multiplier  $A_0^a$ . This is clearly realised by writing the canonical hamiltonian,

$$H_c = \int (\frac{g^2}{64\pi^2} G_{ij}^a G_{ij}^a + \frac{8\pi^2}{g^2} (\pi_i^a + \frac{1}{8\pi} \epsilon_{ij} A^{j,a})^2 - A_0^a \wedge^a) d^2x \quad (42)$$

The physical hamiltonian [22] is now obtained on the constraint surface,

$$H_P = \int (\frac{8\pi^2}{g^2} (\pi_i^a + \frac{1}{8\pi} \epsilon_{ij} A^{j,a})^2 + \frac{g^2}{64\pi^2} G_{ij}^a G_{ij}^a) d^2x \quad (43)$$

Comparing (36) with (43) we see that it is possible to map the  $O(\frac{1}{g^2})$  terms using the identification,

$$(B_i^a + \omega_i^a) \leftrightarrow 4\pi(\pi_i^a + \frac{1}{8\pi} \epsilon_{ij} A^{j,a}) \quad (44)$$

There are two strong reasons for supporting this identification. The first is that it is meaningful to compare only gauge covariant quantities in non-abelian theories since these are related to physical variables. Comparing gauge noncovariant objects - like say potentials - is meaningless since these are unphysical. The mapping (44) is among gauge covariant quantities with the R.H.S. just representing the chromoelectric field  $G_{0i}^a$  (41) in the YMCS theory. The second point is that the mapping is algebraically consistent. Using (10) and (20 - 22), it is easy to show,

$$\{B_i^a(x) + \omega_i^a(x), B_j^b(x') + \omega_j^b(x')\} = 4\pi \delta^{ab} \epsilon_{ij} \delta(x - x') \quad (45)$$

Similarly, using the basic Poisson brackets in the Yang-Mills-Chern-Simons theory [12] it can be shown,

$$\{4\pi(\pi_i^a + \frac{1}{8\pi} \epsilon_{ik} A^{k,a})(x), 4\pi(\pi_j^b + \frac{1}{8\pi} \epsilon_{jl} A^{l,b})(x')\} = 4\pi \delta^{ab} \epsilon_{ij} \delta(x - x') \quad (46)$$

which confirms our observation.

Trying to extend the correspondance we find that the  $O(g^2)$  terms in (36) and (43) do not map on using the identification (44). Thus, in general, the gauge theory to which the Thirring model bosonises is not the Yang-Mills-Chern-Simons theory. It is, however, true in the weak coupling limit

$g^2 \rightarrow 0$  when only the first term in either (36) or (43) survives. We therefore conclude that the SU(2) Thirring model in the weak coupling limit can be mapped on to the SU(2) Yang-Mills-Chern-Simons theory in the limit when the Yang-Mills term vanishes. This result was also obtained earlier [10] in the master lagrangian approach.

We finally extract the bosonisation identities for the Thirring current. Following the usual procedure [6, 8, 9, 10] sources  $S_\mu$  coupled to the current are introduced in the effective action (1),

$$Z = \int d[\psi, \bar{\psi}] \exp i \int d^3x \bar{\psi} (i\partial + m) \psi - \frac{g^2}{2} j_\mu^a j^{\mu a} + \frac{1}{g^2} j_\mu^a S^{\mu, a} \quad (47)$$

where a scaling by  $g^2$  has been done to make  $S_\mu$  dimensionless. Then after the introduction of the auxiliary field,

$$Z = \int d[\psi, \bar{\psi}, B_\mu] \exp i \int d^3x (\bar{\psi} (i\partial + m + \not{B} + \frac{\not{S}}{g^2}) \psi + \frac{1}{2g^2} B^{\mu a} B_\mu^a) \quad (48)$$

Replacing  $B_\mu \rightarrow B_\mu - \frac{S_\mu}{g^2}$  and dropping a non-propagating contact term yields,

$$Z = \int d[\psi, \bar{\psi}, B_\mu] \exp i \int d^3x (\bar{\psi} (i\partial + m + \not{B}) \psi + \frac{1}{2g^2} B^{\mu a} B_\mu^a - \frac{1}{g^4} B_\mu^a S^{\mu a}) \quad (49)$$

Performing the fermionic integration in the large  $m$  approximation leads to the nonabelian self-dual model (4) so that the Thirring current maps to the basic field  $B_\mu^a$  in this model. Since the bosonisation identities are most illuminating in the weak coupling limit  $g^2 \rightarrow 0$ , we henceforth confine our analysis to this regime. The  $B_\mu$  field in (4) maps to the covariant combination  $(B_\mu + \omega_\mu)$  in the embedded gauge theory (13) so that,

$$j_\mu^a \leftrightarrow -\frac{1}{g^2} (B_\mu^a + \omega_\mu^a) \quad (50)$$

Using (44) and (41) it is simple to establish the connection of  $j_i^a$  with the chromoelectric field in the YMCS theory,

$$j_i^a \leftrightarrow -\frac{4\pi}{g^2} (\pi_i^a + \frac{1}{8\pi} \epsilon_{ij} A^{j,a}) = \frac{1}{4\pi} G_{0i}^a \quad (51)$$

The bosonisation rule for  $j_0^a$  cannot be obtained in this way since the operator corresponding to  $(B_0^a + \omega_0^a)$  is not known in the YMCS theory. We therefore take recourse to an alternative route. Instead of evaluating (48) by translating  $B_\mu$ , the fermionic integral can be directly computed to yield,

$$Z = \int dB_\mu \exp i \int d^3x (\mathcal{L} + \frac{1}{8\pi g^2} \epsilon^{\mu\nu\rho} S_\mu^a F_{\nu\rho}^a + \frac{1}{8\pi g^4} \epsilon^{\mu\nu\rho} \epsilon^{abc} B_\mu^a S_\nu^b S_\rho^c) \quad (52)$$

where  $\mathcal{L}$  is the usual lagrangian (5) for the self-dual model and  $F_{\mu\nu}$  is the field tensor (7). The quadratic term in the sources complicates matters for extracting a general bosonisation rule. However, if one is interested in just the correlation functions of  $j_0^a$  then this quadratic piece drops out and we obtain,

$$j_0^a \leftrightarrow \frac{1}{8\pi} \epsilon_{ij} F_{ij}^a \quad (53)$$

Since the basic field  $B_i$  in the self-dual model maps to  $(B_i + \omega_i)$  in the embedded (gauge) theory which in turn can be identified with the combination (44), the bosonisation identity (53) can be recast in terms of the fields in the YMCS gauge theory. Equations (51) and (53) constitute our bosonisation rules which are the analogous of the 1+1 dimensional result of [23].

As concluding remarks we mention that a general method for discussing nonabelian bosonisation (in the low energy regime) of fermionic theories in higher dimensions has been developed. Specifically, it was shown that the SU(2) massive Thirring model, in the leading  $m^{-1}$  approximation, was mapped to a nonabelian gauge theory. The physical sector of this gauge theory, for an arbitrary Thirring coupling  $g^2$ , was explicitly abstracted in the sense that the physical hamiltonian and the Gauss operator were constructed. The present approach, therefore, has an advantage over the master lagrangian approach [10] where it was problematic to obtain the explicit structure of the gauge theory to which the Thirring model bosonised, unless the weak coupling limit  $g^2 \rightarrow 0$  was imposed. Master lagrangians, incidentally, have proved more fruitful in discussing abelian bosonisation [6, 8, 9, 11].

We have next examined the weak coupling limit  $g^2 \rightarrow 0$  in details. In this case it was shown that the corresponding gauge theory simplified to the Yang-Mills Chern-Simons theory in the limit when the Yang-Mills term vanishes. A complete mapping was established by relating a covariant combination of fields in the two gauge theories. Finally, bosonisation identities for the fermionic currents were derived.

It is straightforward to extend our analysis to gauge groups larger than  $SU(2)$ . Then, instead of just the three Euler variables  $\theta, \phi, \psi(20 - 22)$ , more number of additional fields would have to be introduced. Similarly, increasing the space-time dimensionality does not pose difficulties. The evaluation of the fermion determinant can be carried out in a gauge invariant fashion. The non gauge invariant piece, therefore, always comes from the mass term of the auxiliary field. This can be made gauge invariant by the enlargement prescription discussed here and the corresponding gauge theory to which the fermionic model bosonises emerges naturally. As a further future prospect it may be worthwhile to develop a ‘smooth nonabelian bosonisation’ in analogy with the 1+1 dimensional case [3]. Presumably a more general embedding technique has to be envisaged.

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